Mixture states and storage of biased patterns in the Hopfield model: A replica-symmetry-breaking solution

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(Received 11 April 1994)

A learning rule is introduced to store a finite number of biased and an extensive number of unbiased patterns in the Hopfield model. This learning rule enhances the retrieval capacity of the biased patterns and totally suppresses the unwanted symmetric mixture states. Temperature-capacity and capacity-bias phase diagrams are discussed within the one-step replica-symmetry-breaking approach.

PACS number(s): 87.10.+e, 64.60.Cn, 75.10.Hk

I. INTRODUCTION

In general, for the storage and retrieval of correlated patterns in neural networks, the mixture states involving several patterns [1] are important since the state of the network must have a nonzero overlap with all the patterns. The symmetric mixture states represent confusion of the network, i.e., they describe the inability of the network to perceive the details distinguishing the different patterns. Hence they should be totally suppressed. Different methods have been considered that diminish the existence region for these types of states, e.g., a global dynamic constraint [2], or an adjustable uniform field [3]. For finite loading of biased (i.e., statistically independent but effectively correlated) patterns a different learning rule has been proposed [4] allowing no symmetric mixture states.

In these schemes some of the asymmetric mixture states now describe the retrieval behavior of the network. Hence a detailed study of these different mixture states is certainly of interest.

In this paper we consider the loading of a finite number of biased patterns in the presence, however, of an extensive number of unbiased patterns. Within the mean-field theory approach to the Hopfield model, we compare the performance of the Hebb rule with an alternative learning rule allowing no symmetric solutions at all. As mentioned before, asymmetric solutions take over the retrieval function in the latter case. We assume condensation of the biased patterns. It is found that this alternative learning rule significantly enhances the retrieval capacity of these biased patterns. A similar enhancement effect from rewriting the Hebb rule has been discussed in the case of unbiased patterns in [5,6]. For related work on the problem of confusion using a dynamical approach we refer to [7] and the references cited therein.

Guided by the strong reentrance behavior in related replica-symmetric results for the $Q=3$ Potts network [8], and by some replica-symmetric entropy estimates, we especially examine the effect of one-step replica-symmetry breaking (RSB). This effect is substantial and the retrieval capacity for the biased patterns is further enhanced.

The rest of this paper is organized as follows. In Sec. II the model is defined. Section III derives the mean-field free energy for this model within one-step replica-symmetry breaking. In Sec. IV the solutions of the fixed-point equations are discussed and the corresponding temperature-capacity and capacity-phase diagrams are obtained. Finally Sec. V presents some concluding remarks.

II. MODEL

We consider the fully connected Hopfield model

$$H=-\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{N} J_{ij}S_iS_j,\quad S_i=\pm 1 \tag{1}$$

where the synaptic couplings J_{ij} between the binary neurons i and j are given by the learning rule

$$J_{ij}=\frac{1}{N}\sum_{\mu=1}^{\tilde{p}}\left[\xi_i^{\mu}-\frac{v}{\tilde{p}-1}\sum_{\substack{\nu=1\\ \nu\neq\mu}}^{\tilde{p}}\xi_i^{\nu}\right]\left[\xi_j^{\mu}-\frac{w}{\tilde{p}-1}\sum_{\substack{\nu=1\\ \nu\neq\mu}}^{\tilde{p}}\xi_j^{\nu}\right]+\frac{1}{N}\sum_{\mu=\tilde{p}+1}^{p}\xi_i^{\mu}\xi_j^{\mu}\;. \tag{2}$$

The stored patterns $\{\xi^{\mu}\}$, $\mu=1,\ldots,p$, are chosen to be independent random variables. A finite number $\tilde{p}$ of these patterns is allowed to be biased, i.e., those $\{\xi_i^{\mu}\}$, $i=1,\ldots,N;\mu=1,\ldots,\tilde{p}$ take the values ± 1 with probability $P(\xi_i^{\mu}=\pm 1)=(1\pm a)/2$. Here $a\in[-1,1]$ is the bias parameter. Due to the Ising symmetry of the model only the bias interval [0,1] needs to be considered. The other $(p-\tilde{p})$ patterns have zero bias.

The learning rule (2) consists of two parts. The biased patterns are stored according to the first part. For each such pattern the learning rule mixes in all the other biased patterns by subtracting the average over these patterns. As is clear from the second part, the unbiased patterns are stored according to the Hebb learning rule. For

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$v=w=0$ we find the standard Hebb rule. Taking the limit $\tilde{p}\to\infty$ the subtractions $\sum_{\nu\neq\mu}^{\tilde{p}}\xi_i^\nu/(\tilde{p}-1)$ become equal to the bias a by the law of large numbers. Hence in this limit the learning rule (2) with $v=w=1$ and $p=\tilde{p}$ reduces to the standard Hebb rule for biased patterns studied in [2].

III. MEAN-FIELD THEORY: A ONE-STEP REPLICA-SYMMETRY-BREAKING ANALYSIS

Assuming condensation of the $\tilde{p}$ biased patterns and using standard mean-field techniques [1,9], one finds the following expression for the free energy:

$$-\beta f=\lim_{n\to 0}\lim_{N\to\infty}\frac{1}{n}\frac{1}{N}\ln\left\langle\!\!\left\langle 2^{-nN}\mathrm{Tr}\exp\left[\frac{1}{2}\beta\sum_{\lambda=1}^{n}\sum_{\mu=1}^{\tilde{p}}\left[m_{\mu\lambda}-\frac{v}{\tilde{p}-1}\sum_{\substack{\nu=1\\ \nu\neq\mu}}^{\tilde{p}}m_{\nu\lambda}\right]\left[m_{\mu\lambda}-\frac{w}{\tilde{p}-1}\sum_{\substack{\nu=1\\ \nu\neq\mu}}^{\tilde{p}}m_{\nu\lambda}\right]-\frac{1}{2}\alpha\beta nN-\frac{1}{2}\alpha\,\mathrm{Tr}(\ln[\mathbf{q}])\right]\right\rangle\!\!\right\rangle ,\tag{3}$$

with $\lambda=1,\ldots,n$ the replica index, $\alpha=p/N$ the capacity, and $\langle\langle\cdots\rangle\rangle$ denoting the average over the $\tilde{p}$ biased patterns. Here the following order parameters are introduced:

$$m_{\mu\lambda}=\frac{1}{N}\sum_{i=1}^{N}\langle\langle\xi_i^\mu\langle S_i^\lambda\rangle\rangle\rangle ,\tag{4}$$

$$q_{\lambda\lambda'}=\frac{1}{N}\sum_{i=1}^{N}\langle\langle\langle S_i^\lambda\rangle\langle S_i^{\lambda'}\rangle\rangle\rangle,\quad \lambda\neq\lambda'\tag{5}$$

where $\langle\cdots\rangle$ stands for the thermal average. So the $m_{\mu\lambda}$ describe the macroscopic overlap with the condensed patterns and the $q_{\lambda\lambda'}$ are the Edwards-Anderson order parameters.

Using the one-step RSB scheme with the following assumption for the order parameters $m_{\mu\lambda}$ and $q_{\lambda\lambda'}$ [10–12]:

$$m_{\mu\lambda}=m_\mu,\quad q_{\lambda\lambda'}=q_{\beta_1\beta_2}^{\alpha_1\alpha_2}=\begin{cases}q_1 & \text{if } \alpha_1=\beta_1\\ q_0 & \text{if } \alpha_1\neq\beta_1,\end{cases}\tag{6}$$

where $\alpha_1,\beta_1=1,\ldots,n/k,\alpha_2,\beta_2=1,\ldots,k$, and $0\le k\le n$, we obtain after some tedious but standard calculations the following expression for the free energy per neuron:

$$\begin{aligned}f(\mathbf{m},q_0,q_1,k,\beta)=&\frac{1}{2}\sum_{\mu=1}^{\tilde{p}}\left[m_\mu-\frac{v}{\tilde{p}-1}\sum_{\substack{\nu=1\\ \nu\neq\mu}}^{\tilde{p}}m_\nu\right]\left[m_\mu-\frac{w}{\tilde{p}-1}\sum_{\substack{\nu=1\\ \nu\neq\mu}}^{\tilde{p}}m_\nu\right]\\ &+\frac{\alpha}{2}\left\{\frac{1}{\beta}\ln(1-C_1)+\frac{1}{k\beta}\ln\left[1-\frac{k\beta(q_1-q_0)}{1-C_1}\right]-\frac{q_0}{1-C_1-k\beta(q_1-q_0)}+1\right\}\\ &-\frac{k\beta}{2}(q_0^2r_0-q_1r_1)-\frac{1}{2}r_1C_1-\frac{1}{k\beta}\left\langle\!\!\left\langle\int Dz_1\ln\{\int Dz_2\cosh^k(\beta H(\xi;z_1,z_2))\}\right\rangle\!\!\right\rangle ,\end{aligned}\tag{7}$$

with

$$\begin{aligned}H(\xi;z_1,z_2)=&z_1\sqrt{q_0r_0}+z_2\sqrt{r_1-r_0q_0}\\ &+\sum_{\nu=1}^{\tilde{p}}\left[m_\nu-\frac{v+w}{\tilde{p}-1}\sum_{\substack{\gamma=1\\ \gamma\neq\nu}}^{\tilde{p}}m_\gamma+\frac{vw}{(\tilde{p}-1)^2}\sum_{\substack{\gamma=1\\ \gamma\neq\nu}}^{\tilde{p}}\sum_{\substack{\mu=1\\ \mu\neq\gamma}}^{\tilde{p}}m_\mu\right]\xi^\nu ,\end{aligned}\tag{8}$$

$$Dz=dz(2\pi)^{-1/2}\exp(-z^2/2) ,\tag{9}$$

and

$$r_0=\frac{\alpha}{[1-C_1-k\beta(q_1-q_0)]^2} ,\tag{10}$$

$$r_1=r_0q_0+\frac{\alpha(q_1-q_0)}{(1-C_1)[1-C_1-k\beta(q_1-q_0)]} ,\tag{11}$$

where $C_1=\beta(1-q_1)$.

The extrema of the free energy (7) can be found by studying the fixed-point equations for m_μ, q_0, and C_1, together with the saddle-point equation obtained by taking the derivative of f with respect to k. Since the way to derive these formulas is standard and since their explicit expressions are algebraically complicated we do not write them down.

For arbitrary values of v and w these fixed-point equations allow the following solutions: asymmetric solutions $\mathbf{m}=(m_1,m_{\tilde{p}-1},\ldots,m_{\tilde{p}-1}),q_1,q_0\neq 0$, which have retrieval properties $(m_1 \gg m_{\tilde{p}-1})$; symmetric solutions $\mathbf{m}=(m_{\tilde{p}},\ldots,m_{\tilde{p}}),q_1,q_0\neq 0$ representing confusion in the network; spin-glass solutions $\mathbf{m}=0,q_1,q_0\neq 0$ and a paramagnetic solution $\mathbf{m}=0,q_1=q_0=0$. There are no symmetric solutions of the form $\mathbf{m}=(m_n,\ldots,m_n,0,\ldots,0)$ with $n\leq\tilde{p}$, and hence no Mattis solutions. For the specific values $v=w=1, v=1, w=0$ and $v=0, w=1$, the unwanted symmetric solutions do not exist.

In the following section we analyze these fixed-point equations numerically. First we discuss the results for the replica-symmetric approximation to this network. The corresponding free energy is obtained by taking the limit $k\to 0$ and $q_1\to q_0$ [10]. Guided by some related results for the $Q=3$ Potts model [8] where the capacity for $v=w=1$ is always larger than the capacity for $v=1,w=0$ and the one for $v=0,w=1$ we will restrict ourselves to the case $v=w=1$ and its comparison with the Hebb rule ($v=w=0$). Secondly we explain the one-step RSB results.

IV. RETRIEVAL PROPERTIES AND PHASE DIAGRAMS

A. Replica-symmetric results at zero temperature

For the asymmetrical retrieval solutions the fixed-point equations for m_μ and q_0 can be further reduced at $T=0$ to only one fixed-point equation in $y=\tilde{p}(m_1-m_{\tilde{p}-1})/(\tilde{p}-1)\sqrt{2q_0 r_0}$. An expansion of this equation in terms of y and inspection of the sign of the coefficient of the third order term teaches us that the order of the transition from the asymmetrical retrieval phase to the spin-glass phase depends on the number of biased patterns $\tilde{p}$ and the bias parameter a. For $\tilde{p}\leq 4$ the transition is always second order. For $\tilde{p}=5$ it is first order if $a\in[0,\sqrt{11/45})$ and second order if $a\in[\sqrt{11/45},1]$, and for $\tilde{p}\geq 6$ it is always first order. The second order transition occurs at

$$\alpha=\frac{2}{\pi}\left[\left[\frac{\tilde{p}}{\tilde{p}-1}\right]^2(1-a^2)-1\right]^2 \quad \text{for } a<\left[1-\left[\frac{\tilde{p}-1}{\tilde{p}}\right]^2\right]^{1/2}. \qquad (12)$$

This condition implies that in the limit $\alpha\to 0$ retrieval is only possible in a restricted region of the interval $a\in[0,1]$. On the other hand, a standard signal-to-noise ratio analysis predicts retrieval in the whole bias interval [0,1]. This at first sight contradictory result is explained by the fact that we have also found asymmetric retrieval solutions which exist for all $a\in[0,1]$. However, they are unstable.

Looking at Table I, we can compare both the retrieval capacities of the $v=w=1$ model and the $v=w=0$ model. The latter has been studied in [13]. For $a=0$ the $v=w=0$ model reduces to the standard Hopfield model such that we find a maximal storage capacity $\alpha_c=0.1379$ for all values of $\tilde{p}$. This is in contrast with the $v=w=1$ model. As can be read off from the first column, the latter has a retrieval capacity larger than 0.1379 for $\tilde{p}\leq 7$, whereas for $\tilde{p}>7$ it is somewhat smaller, attaining a minimal value for $\tilde{p}=12$. This does not mean that the overall storage capacity of the network is increased but it indicates that the biased patterns $\tilde{p}$ can still be retrieved when the network is loaded with a total number of patterns greater than $0.1379N$. Asymptotically, for large $\tilde{p}$, the retrieval capacity converges to the value $\alpha_c=0.1379$. This behavior is changed by introducing a bias. In this case the retrieval capacity of the $v=w=1$ model exceeds that of the $v=w=0$ model, except in the interval $[a_1,a_2]$, indicated in the second column.

TABLE I. The capacity α_c for the Hopfield network with $v=w=1$, $\tilde{p}$ biased, and $p-\tilde{p}$ unbiased patterns for $a=0$; the bias interval $[a_1,a_2]$ where this model has a smaller capacity than the $v=w=0$ model and the maximal bias a_c above which no stable retrieval solutions exist.

$\tilde{p}$	$\alpha_c(a=0)$	$[a_1,a_2]$	$a_c(v=w=1)$	$a_c(v=w=0)$
2	5.7296	[0.842,1]	0.866	1
3	0.9947		0.754	0.707
4	0.3851		0.661	0.577
5	0.2130		0.599	0.500
6	0.1621	[0.154,0.351]	0.552	0.447
7	0.1430	[0.068,0.335]	0.523	0.408
8	0.1345	[0,0.317]	0.508	0.378
9	0.1303	[0,0.300]	0.535	0.354
10	0.1283	[0,0.285]	0.549	0.333
100	0.1356	[0,0.090]	0.871	0.101

For both models there exists a maximal bias a_c above which no stable asymmetrical retrieval solutions exist. These values are given in the third and fourth columns of Table I. We find that for $\tilde{p}\geq 3$, $a_c(v=w=1)>a_c(v=w=0)$, indicating that for the $v=w=1$ model stable asymmetrical retrieval solutions exist in a larger bias region. We note that for $\tilde{p}=2$, 3, 4, and 5 (in the second order regime) the values of $\alpha_c(v=w=1)$ and a_c correspond with (12).

Finally we remark that in the limit $\tilde{p}\to\infty$ the storage capacity does only converge to the value corresponding to the Hebb rule for biased patterns [2] when $p=\tilde{p}$. Otherwise, for $p>\tilde{p}$, it is smaller for all values of $a\neq 0$, e.g., for $a=0.4$ we find 0.0314 versus 0.0481.

B. Replica-symmetric results at finite temperature

Let us now turn to the replica-symmetric results for $T\neq 0$. Some typical T-α phase diagrams are presented in Figs. 1 and 2. The line T_g (dash-dotted line) indicates the transition from the spin-glass solutions to the disordered paramagnetic solution. This transition is second order and occurs at $T_g=1+\sqrt{\alpha}$. It is identical to the one in the Hopfield model with the Hebb rule and zero bias [1]. In the region bounded by the line T_A (solid and dashed

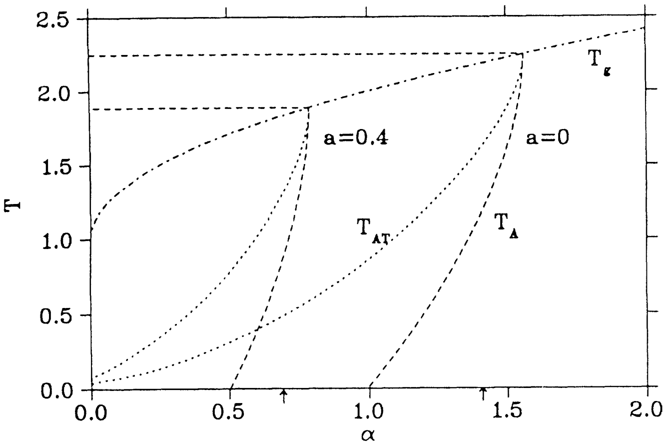

FIG. 1. The T-α diagram for the Hopfield network with $v=w=1$, $\tilde{p}=3$ biased, and $p-\tilde{p}$ unbiased patterns for $a=0$ and 0.4. The meaning of the curves is explained in the text.

lines) asymmetric retrieval solutions exist and they are the local minima of the free energy. The transition from the asymmetric retrieval solutions to the paramagnetic solution is second order (dashed line). The transition temperature reads

$$T_A=(1-a^2)\left[\frac{\tilde{p}}{\tilde{p}-1}\right]^2 \quad \text{for } 0\le\alpha\le\left[(1-a^2)\left[\frac{\tilde{p}}{\tilde{p}-1}\right]^2-1\right]^2 . \tag{13}$$

Finally the transition from the retrieval solution to the spin-glass solution can be first or second order (solid, respectively, dashed line). For $\tilde{p}\le 4$ it is second order for all temperatures. From $\tilde{p}=5$ onwards it starts being first order at low temperatures. For $\tilde{p}\to\infty$ it is first order at all temperatures.

Figure 1 shows that the introduction of a bias leads to a smaller retrieval region. However, the overall shape of the T_A line remains the same. From Fig. 2 one learns how the T_A line and hence the retrieval region behaves with increasing $\tilde{p}$. Asymptotically, for $\tilde{p}\to\infty$ the T_A line resembles the Mattis transition line in the Hopfield model [1]. Compared with the $v=w=0$ rule [13] the retrieval region in the T-α plane is always larger in the present model.

From both phase diagrams we also learn that α_c increases for all temperatures for $\tilde{p}<6$. From $\tilde{p}=6$ onwards it only increases at low temperatures and in the neighborhood of the maximal transition temperature T_A. We expect this behavior in these regions to be a consequence of RSB [14].

To check the stability of the replica-symmetric approximation we have computed the de Almeida-Thouless (AT) line [15]. As a result we find that below the dotted lines T_{AT} shown in Figs. 1 and 2, replica symmetry is broken. It is interesting to note that for $\alpha\to 0$ the temperature T_{AT} is given by

$$T_{AT}\simeq\frac{4}{3\sqrt{2\pi}}A\left[\left[\frac{2}{\pi}\right]^{1/2}A+\sqrt{\alpha}\right], \tag{14}$$

$$A=\left\langle\!\!\left\langle \exp\left[-y^2\left[\xi^1-\frac{1}{\tilde{p}-1}\sum_{\mu=2}^{\tilde{p}}\xi^\mu\right]^2\right]\right\rangle\!\!\right\rangle . \tag{15}$$

One finds as numerical value, e.g., $A=0.25$ for $\alpha=0$, $a=0$, $\tilde{p}=3$. This quantity A becomes smaller with in-

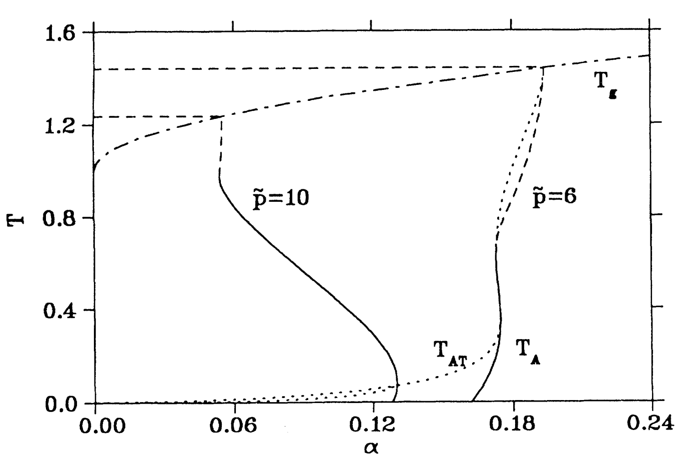

FIG. 2. The T-α diagram for the Hopfield network with $v=w=1$, $\tilde{p}=6,10$ biased, and $p-\tilde{p}$ unbiased patterns for $a=0$. The meaning of the curves is explained in the text.

creasing $\tilde{p}$. For the Hopfield model, however, T_{AT} decreases to zero exponentially with α [1].

The line T_{AT} always intersects the retrieval phase boundary T_A at the point where the gradient of T_A becomes infinite. This implies, as conjectured in [14], that the deflection of the T_A line is a consequence of RSB. As Fig. 1 indicates, the retrieval region below the T_{AT} line is large for $\tilde{p}=3$ and the discrepancy between the maximal value of α_c and its value at $T=0$ is also large. For $\tilde{p}=6$ and 10 (see Fig. 2) the retrieval region of broken replica symmetry and the difference between the maximal retrieval capacity at finite T and its value at $T=0$ is much smaller.

C. One-step replica-symmetry breaking

It is clear by now that the assumption of replica symmetry is far from being justified in the retrieval region below the T_{AT} line. Therefore we have performed a one-step RSB calculation of the capacity at zero temperature, starting from Eqs. (7)–(11).

Some results are shown in Figs. 1 and 3. In Fig. 1 the arrows indicate the one-step RSB result for the $\tilde{p}=3$ capacity at $T=0$. Compared with the replica-symmetric capacity values, e.g., 0.9947 for $a=0$ and 0.5043 for $a=0.4$ [recall (12)], we now find the values 1.3864 (respectively, 0.7002). So, the effect of taking into account one-step RSB is, clearly, that the capacity increases substantially towards the maximal value, i.e., 1.5625 (respectively, 0.7291) [recall (13)], at the maximal transition temperature T_A. This is also seen in Fig. 3 where the capacity of the asymmetrical retrieval states is plotted as a function of the bias parameter a for $\tilde{p}=2,3,4$ biased and $p-\tilde{p}$ unbiased patterns. The dotted line represents the replica-symmetric result and the solid line follows from the one-step RSB approach. The effect of one-step RSB becomes smaller with increasing $\tilde{p}$ and with increasing a. For all finite $\tilde{p}$ and even in the limit $\tilde{p}\to\infty$ we find a one-step RSB value for α_c smaller than its maximal value at finite temperature. In particular, for the Hopfield model we verified the result $\alpha_c=0.138\,186\,49$ found in [16].

In the limit $\alpha\to 0$ we find again that the stable asymmetric retrieval solutions occur in a restricted region $[0,a_c]$ of the interval [0,1]. The value of a_c obtained here is the same as the replica-symmetric value.

To get an idea about the stability with respect to further breaking, we have calculated the $T=0$ entropy at α_c. We find, e.g., the following results: for $\tilde{p}=3, S(a=0)=-0.1676\times 10^{-1}$, $S(a=0.4)=-0.1682\times 10^{-1}$ and for $\tilde{p}=10, S(a=0)=-0.2505\times 10^{-3}$. These values are still negative, but much smaller than their analogs in the replica-symmetric approximation. Indeed for $\tilde{p}=3$ the latter read $S(a=0)=-0.1055$ and $S(a=0.4)=-0.9341\times 10^{-1}$, while for $\tilde{p}=10$ we get $S(a=0)=-0.6699\times 10^{-2}$.

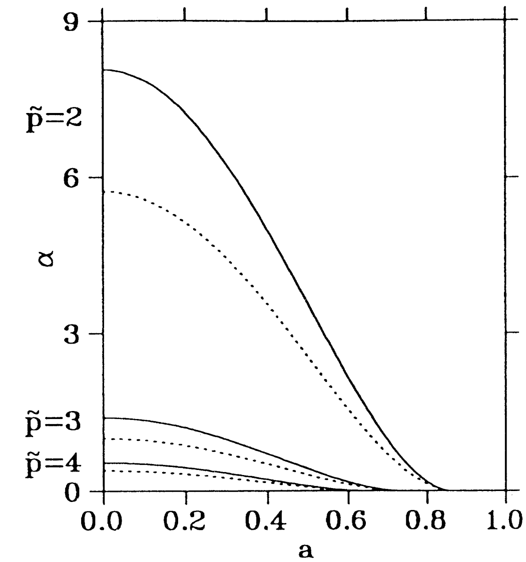

FIG. 3. The capacity α as a function of a for the Hopfield network with $v=w=1$, $\tilde{p}=2,3,4$ biased, and $p-\tilde{p}$ unbiased patterns. The meaning of the curves is explained in the text.

V. CONCLUDING REMARKS

In conclusion, we have discussed an alternative learning rule for the storage and retrieval of a finite number of biased patterns in the presence of an extensive number of unbiased patterns in the Hopfield model. In the replica-symmetric approximation the retrieval capacity for the biased patterns is enhanced without effectively damaging the ability of the network to retrieve the other patterns. The unwanted symmetric mixture states representing confusion are absent.

The structure of the T-α phase diagram and an explicit calculation of the AT line indicates a strong RSB effect. Performing a one-step RSB calculation we find that the retrieval capacity of these biased patterns is even further enhanced by a substantial amount, especially for a small number of biased patterns.

Our results support the conjecture that the full RSB retrieval phase line may be found by drawing a line vertically from the intersection of the AT line with the retrieval phase boundary to the zero-temperature axis.

ACKNOWLEDGMENTS

This work has been supported in part by the Research Fund of the K. U. Leuven (Grant No. OT/91/13). The authors are indebted to G. M. Shim for stimulating discussions and to R. Kühn and H. Steffan for useful correspondence on the results of [16]. They also would like to thank the Belgian National Fund for Scientific Research and the Inter-University Institute for Nuclear Sciences for financial support.

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